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Best L_1-Approximation on Finite Point Sets: Rate of Convergence

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1. INTRODUCTION

Let $L_1[0, 1]$ be the space of real-valued Lebesgue integrable functions on $[0, 1]$ with norm $\|f\|_1 = \int_0^1 |f(x)|\, dx$; $C[0, 1] \subset L_1[0, 1]$ the subspace of continuous functions; $U_n \subset C[0, 1]$ an n-dimensional Haar subspace.

By the classical theorem of Jackson [3], for any $f \in C[0, 1]$ there exists a unique polynomial $p_n(f) \in U_n$ such that

$$\|f - p_n(f)\|_1 = \inf_{q_n \in U_n} \|f - q_n\|_1. \tag{1}$$

In what follows, $p_n(f)$ will always denote the best L_1-approximation to $f \in C[0, 1]$.

The usual approach to the L_1-approximation problem consists in replacing in (1) the L_1-norm by a discrete L_1-norm.

Let $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$ be a discrete set of points on $[0, 1]$: $\Delta x_i = x_{i+1} - x_i$, $x_i^* = (x_i + x_{i+1})/2$, $i = \overline{0, N}$ (here and throughout the paper we use the abbreviation $\overline{n, m} = \{n, n+1, ..., m-1, m\}$), $\delta = \max_{0 \le i \le N} \Delta x_i$. Then we can define the discrete L_1-norm by

$$\|f\|_{1,\delta} = \sum_{i=0}^{N} |f(x_i^*)|\, \Delta x_i \tag{2}$$

and look for solutions of the L_1-approximation problem for this (semi)norm:

$$\|f - p_n(f)_\delta\|_{1,\delta} = \inf_{q_n \in U_n} \|f - q_n\|_{1,\delta}. \tag{3}$$

The best discrete L_1-approximation $p_n(f)_\delta$ is not unique in general. We shall denote by $Y_n(f)_\delta$ the set of polynomials $p_n(f)_\delta$ satisfying (3). A detailed discussion of best discrete L_1-approximation can be found in Rice [8] and

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Rivlin [9]. In [9] it is shown that the solution of (3) can be obtained as a solution of a linear programming problem.

It is natural to expect that for $f \in C[0, 1]$ all $p_n(f)_\delta$ tend to $p_n(f)$ as $\delta \to 0$, i.e.,

$$\sup_{p_n(f)_\delta \in Y_n(f)_\delta} \| p_n(f) - p_n(f)_\delta \|_C \to 0 \qquad (\delta \to 0),$$

where $\|\cdot\|_C$ is the supremum norm. This result was first proved by Motzkin and Walsh [6]. (It also follows from a general theorem of Kripke [4].)

In the present paper we shall be interested in the rate of convergence of $p_n(f)_\delta$ to $p_n(f)$ as $\delta \to 0$. This problem was attacked by Usow [10]. Set $\mathrm{Lip}_M\, \alpha = \{f \in C[0, 1]: \omega_f(h) \leqslant Mh^\alpha\}$, where $\omega_f(h) = \sup_{|x_1 - x_2| \leqslant h} |f(x_1) - f(x_2)|$; $M > 0$, $0 < \alpha \leqslant 1$; and let $\{\varphi_i\}_{i=1}^n$ be a basis in U_n. Usow [10] has shown that if f and φ_i $(1 \leqslant i \leqslant n)$ belong to $\mathrm{Lip}_M\, 1$ and the set of zeros of $f - p_n(f)$ is of measure zero and contains at least n isolated points, then

$$\sup_{p_n(f)_\delta \in Y_n(f)_\delta} \| p_n(f) - p_n(f)_\delta \|_C = O(\sqrt{\delta}), \tag{4}$$

where the constant in O depends only on f and U_n.

The question of sharpness of the estimation (4) remained open.

Our principal result is that for a wide class of functions

$$\sup_{p_n(f)_\delta \in Y_n(f)_\delta} \| p_n(f) - p_n(f)_\delta \|_C = O(\omega_f(\delta)) \tag{5}$$

and this rate of convergence is the best possible in general. Evidently, (5) is a strong improvement of (4). (It is interesting to observe that the rate of convergence in discretization of Čebysev approximation is also $\omega_f(\delta)$ (see [1, p. 92]).)

2. New Results

In what follows $\{\varphi_i\}_{i=1}^n$ will always be a basis in U_n and we assume that $\varphi_1 \equiv 1$ and $\varphi_i \in \mathrm{Lip}_{M^*}\, 1$ $(2 \leqslant i \leqslant n)$ for some $M^* > 0$.

We start with a generalization of Usow's theorem.

THEOREM 1. *If $f \in \mathrm{Lip}_M\, \alpha$ and $0 < \delta < c_1(\alpha, M, U_n)$, then*

$$\sup_{p_n(f)_\delta \in Y_n(f)_\delta} \| p_n(f) - p_n(f)_\delta \|_C \leqslant c_2(\alpha, M, U_n)\, \delta^{\alpha^2/(\alpha+1)}, \tag{6}$$

where the constants $c_j(\alpha, M, U_n)$ $(j = 1, 2)$ depend only on α, M and U_n.

Theorem 1 is a generalization of Usow's result because it does not impose any restriction on the set of zeros of $f-p_n(f)$ and estimation (6) is uniform on the class of functions $\mathrm{Lip}_M\,\alpha$. But from the point of view of the rate of convergence it does not improve (4), because the proof of Theorem 1 goes essentially along the same lines as that of (4).

Our main result is the following

THEOREM 2. *Let $f\in C[0,1]$ and φ_i $(1\leqslant i\leqslant n)$ be twice continuously differentiable. If $f-p_n(f)$ has a finite number of zeros and δ is small enough, then*

$$\sup_{p_n(f)_\delta\in Y_n(f)_\delta}\|p_n(f)-p_n(f)_\delta\|_C\leqslant c_3(f,U_n)\,\omega_f(\delta), \tag{7}$$

where the constant $c_3(f,U_n)$ depends only on f and U_n. Moreover estimation (7) cannot be improved in general.

Remark 1. The condition $f\in\mathrm{Lip}_M\,\alpha$ is not essential in Theorem 1. An estimation for the rate of convergence also can be given in case $f\in C[0,1]$. But in the general case, the order of convergence cannot be obtained explicitly; it will depend on ω_f.

Remark 2. In general the discrete L_1-norm of f can be defined by $\sum_{i=0}^{N}|f(\xi_i)|\,\Delta x_i$, where $\xi_i\in[x_i,x_{i+1})$ are arbitrary fixed points. In particular Usow considered the case $\xi_i=x_i$ but his proof still goes for any ξ_i. Theorem 1 remains also true when ξ_i are arbitrary, but in the proof of Theorem 2 the choice of ξ_i to be the middle point of the interval $[x_i,x_{i+1}]$ is essential.

Remark 3. The proof of Theorem 1 is based on a standard method, applying a strong unicity type result. This method was used by Cheney ([1, p. 92]) in the case of Čebysev approximation, by Peetre [7] for L_p-approximation $(1<p\leqslant\infty)$ and by Usow [10] for L_1-approximation. In contrast with L_1-approximation, this standard method gives sharp estimations in discretization of Čebysev approximation. The proof of Theorem 2 which gives already the best possible estimation for the rate of convergence of discrete L_1-approximation is based on more delicate considerations connected with specific features of approximation in L_1-norm.

3. Proof of Theorem 1

We shall need some simple propositions. In what follows $c_i(\cdots)$ will denote constants depending only on quantities specified in the brackets.

PROPOSITION 1. *For any $q_n \in U_n$ and $0 < h \leqslant 1$*

$$\omega_{q_n}(h) \leqslant c_4(U_n) \|q_n\|_1 h, \tag{8}$$

where we may assume that $c_4(U_n) \geqslant 1$.

Proof. Using that $\varphi_i \in \mathrm{Lip}_{M^*} 1$ $(1 \leqslant i \leqslant n)$ and the equivalence of norms in finite-dimensional spaces we have for $q_n = \sum_{i=1}^n a_i \varphi_i$

$$\omega_{q_n}(h) \leqslant \sum_{i=1}^n |a_i| \, \omega_{\varphi_i}(h) \leqslant M^* h \sum_{i=1}^n |a_i| \leqslant c_4(U_n) \|q_n\|_1 h.$$

PROPOSITION 2. *For any $q_n \in U_n$ and $0 < \delta \leqslant 1/c_4(U_n)$*

$$\|q_n\|_1 \leqslant 2 \|q_n\|_{1,\delta}, \tag{9}$$

$$\|q_n\|_{1,\delta} \leqslant \tfrac{3}{2} \|q_n\|_1. \tag{10}$$

Proof. Obviously, for any $f \in C[0, 1]$

$$| \|f\|_1 - \|f\|_{1,\delta}| \leqslant \omega_f(\delta/2). \tag{11}$$

Hence and by (8), we have

$$| \|q_n\|_1 - \|q_n\|_{1,\delta}| \leqslant \omega_{q_n}\left(\frac{\delta}{2}\right) \leqslant \frac{c_4(U_n)\delta}{2} \|q_n\|_1 \leqslant \frac{\|q_n\|_1}{2}.$$

This immediately implies (9) and (10).

PROPOSITION 3. *Let $f \in C[0, 1]$, $p_n(f)_\delta \in Y_n(f)_\delta$, where $0 < \delta \leqslant 1/c_4(U_n)$. Then for any $0 < h \leqslant 1$*

$$\omega_{p_n(f)}(h) \leqslant c_5(U_n) \, \omega_f(h), \tag{12}$$

$$\omega_{p_n(f)_\delta}(h) \leqslant c_6(U_n) \, \omega_f(h). \tag{13}$$

Proof. Set $\bar{f}(x) = f(x) - f(0)$. Evidently $p_n(\bar{f}) = p_n(f) - f(0)$, $p_n(f)_\delta - f(0) = p_n(\bar{f})_\delta \in Y_n(\bar{f})_\delta$. Therefore by (8), we have $\omega_{p_n(f)}(h) \equiv \omega_{p_n(\bar{f})}(h) \leqslant c_4(U_n) \|p_n(\bar{f})\|_1 h \leqslant 2c_4(U_n) \|\bar{f}\|_1 h \leqslant 2c_4(U_n) \|\bar{f}\|_C h \leqslant 2c_4(U_n) \, \omega_f(1)h \leqslant 4c_4(U_n) \, \omega_f(h)$ (here we used the inequality $\omega_f(1)h \leqslant 2\omega_f(h)$). Further by (8), (9) and (11), we obtain

$$\begin{aligned}
\omega_{p_n(f)_\delta}(h) \equiv \omega_{p_n(\bar{f})_\delta}(h) &\leqslant c_4(U_n) \| p_n(\bar{f})_\delta\|_1 h \\
&\leqslant 2c_4(U_n) \| p_n(\bar{f})_\delta \|_{1,\delta} h \leqslant 4c_4(U_n) \|\bar{f}\|_{1,\delta} h \\
&\leqslant 4c_4(U_n)(\|\bar{f}\|_1 + \omega_{\bar{f}}(\delta)) h \\
&\leqslant 8c_4(U_n) \, \omega_f(1)h \leqslant 16c_4(U_n) \, \omega_f(h).
\end{aligned}$$

PROPOSITION 4. *Let* $f \in C[0,1]$, $p_n(f)_\delta \in Y_n(f)_\delta$, *where* $0 < \delta \leqslant 1/c_4(U_n)$. *Then*

$$\|f - p_n(f)_\delta\|_1 \leqslant \|f - p_n(f)\|_1 + c_7(U_n)\,\omega_f(\delta). \tag{14}$$

Proof. Equation (11) and Proposition 3 imply

$$\begin{aligned}\|f - p_n(f)_\delta\|_1 &\leqslant \|f - p_n(f)_\delta\|_{1,\delta} + \omega_f(\delta) + \omega_{p_n(f)_\delta}(\delta)\\ &\leqslant \|f - p_n(f)\|_{1,\delta} + (1 + c_6(U_n))\,\omega_f(\delta)\\ &\leqslant \|f - p_n(f)\|_1 + \omega_f(\delta) + \omega_{p_n(f)}(\delta) + (1 + c_6(U_n))\,\omega_f(\delta)\\ &\leqslant \|f - p_n(f)\|_1 + c_7(U_n)\,\omega_f(\delta).\end{aligned}$$

The following strong unicity type theorem is proved in [5, Theorem 2]:

Let $f^* \in C[0,1]$, $p_n(f^*) \equiv 0$ and $\omega_{f^*}(h) \leqslant \omega(h)$, where $\omega(h)$ is a fixed modulus of continuity. Then

$$\begin{aligned}\sup\{\|q_n\|_1 : q_n \in U_n, \|f^* - q_n\|_1 \leqslant \|f^*\|_1 + 2\varepsilon\}\\ \leqslant c_8(\omega, U_n)\, I_\omega(\varepsilon),\end{aligned} \tag{15}$$

where $I_\omega(\varepsilon)$ is the inverse of $S_\omega(\varepsilon) = \int_0^{\omega^{-1}(\varepsilon)} (\varepsilon - \omega(t))\,dt$ and $\varepsilon > 0$ is an arbitrary real for which $I_\omega(\varepsilon)$ is defined, i.e., $0 < \varepsilon \leqslant \int_0^1 (\omega(1) - \omega(t))\,dt$.

For $f \in \mathrm{Lip}_M\,\alpha$ set $f^* = f - p_n(f)$; $q_n = p_n(f)_\delta - p_n(f)$, where $p_n(f)_\delta \in Y_n(f)_\delta$ and $0 < \delta < 1/c_4(U_n)$. Then $p_n(f^*) \equiv 0$ and by (12), $\omega_{f^*}(h) \leqslant c_9(M, U_n)\,h^\alpha$. Setting $\omega(h) = c_9(M, U_n)\,h^\alpha$ we obtain $I_\omega(\varepsilon) = c_{10}(\alpha, M, U_n)\,\varepsilon^{\alpha/(\alpha+1)}$. Further by (14)

$$\begin{aligned}\|f^* - q_n\|_1 - \|f^*\|_1 &= \|f - p_n(f)_\delta\|_1 - \|f - p_n(f)\|_1\\ &\leqslant M c_7(U_n)\,\delta^\alpha.\end{aligned}$$

Hence and by (15) for any $0 < \delta < c_{11}(\alpha, M, U_n)$

$$\begin{aligned}\|p_n(f) - p_n(f)_\delta\|_C &\leqslant c_{12}(U_n)\,\|p_n(f) - p_n(f)_\delta\|_1\\ &= c_{12}(U_n)\,\|q_n\|_1 \leqslant c_{13}(\alpha, M, U_n)\,\delta^{\alpha^2/(\alpha+1)}.\end{aligned}$$

Theorem 1 is proved.

4. PROOF OF THEOREM 2

We start with verifying the upper bound of Theorem 2.

Evidently we may assume that $p_n(f) \equiv 0$ and f has finite number of zeros. Let $0 < t_1 < \cdots < t_m < 1$ be all the zeros of f inside $(0,1)$. Set $t_0 = 0$.

$t_{m+1}=1$ (t_0 and t_{m+1} may also be zeros of f) and $t=\min_{0\leqslant i\leqslant m}(t_{i+1}-t_i)$. Throughout the rest of the proof we assume that $\delta<\min\{t/4,\ 1/c_\Delta(U_n)\}$.

For any $k=\overline{0, m+1}$ set

$$i_0=0;\qquad i_k=\max\left\{j: x_j\leqslant t_k-\frac{\delta}{2}\right\}\qquad (k=\overline{1, m+1}),$$

$$s_k=\min\left\{j: x_j\geqslant t_k+\frac{\delta}{2}\right\}\qquad (k=\overline{0, m});\ s_{m+1}=N+1.$$

Evidently,

$$\begin{aligned}&\frac{\delta}{2}\leqslant t_k-x_{i_k}<\frac{3\delta}{2}\qquad (k=\overline{1, m+1}),\\ &\frac{\delta}{2}\leqslant x_{s_k}-t_k<\frac{3\delta}{2}\qquad (k=\overline{0, m});\end{aligned}\tag{16}$$

hence $x_{s_k}<x_{i_{k+1}}$ $(k=\overline{0, m})$ and

$$\frac{\delta}{2}\leqslant x_{s_k}-x_{i_k}<3\delta\qquad (k=\overline{0, m+1}).\tag{17}$$

Further by $\{t_k\}_{k=0}^{m+1}$ and $\{x_i\}_{i=0}^{N+1}$ we define a linear operator D acting from $C[0, 1]$ into $L_\infty[0, 1]$. For $g\in C[0, 1]$

$$\begin{aligned}D(g)&=g, &&x\in[x_{i_k}, x_{s_k}]\ (k=\overline{0, m+1}),\\ &=g(x_i^*), &&x\in[x_i, x_{i+1}]\ (i=\overline{0, N};\ i\neq i_k, s_k-1;\ k=\overline{0, m+1}).\end{aligned}\tag{18}$$

Obviously for any $x\in[0, 1]$, we have

$$\operatorname{sign} f=\operatorname{sign} D(f).\tag{19}$$

We shall establish some properties of D.

PROPOSITION 5. *Let $g\in C[0, 1]$, $1\leqslant k\leqslant m$. Then for any $x\in[0, 1]$*

$$|D(g, x)-g(t_k)|\leqslant 2\omega_g(|x-t_k|).\tag{20}$$

Proof. Assume, e.g., that $x\geqslant t_k$. Then for $x\in[t_k, x_{s_k}]$, $D(g, x)=g(x)$; thus (20) is evident. Further if $x\geqslant x_{s_k}$ then $x\in[x_r, x_{r+1}]$ for some r and therefore $D(g, x)$ is equal to $g(x)$ or $g(x_r^*)$. Thus by (16)

$$\begin{aligned}|D(g, x)-g(t_k)|&\leqslant|g(x)-g(t_k)|+|D(g, x)-g(x)|\\ &\leqslant\omega_g(x-t_k)+\omega_g(\delta/2)\\ &\leqslant\omega_g(x-t_k)+\omega_g(x_{s_k}-t_k)\leqslant 2\omega_g(x-t_k).\end{aligned}$$

For $x \leqslant t_k$ the proof can be obtained analogously.

LEMMA 1. *For any $q_n \in U_n$, we have*

$$\left| \int_0^1 D(q_n) \operatorname{sign} D(f)\, dx \right| \leqslant c_{14}(U_n) \|q_n\|_C \, \delta^2. \tag{21}$$

Proof. By our assumption $p_n(f) \equiv 0$ and f has a finite number of zeros. Then, by a well-known theorem and (19), for any $q_n \in U_n$

$$\int_0^1 q_n \operatorname{sign} D(f)\, dx = \int_0^1 q_n \operatorname{sign} f\, dx = 0. \tag{22}$$

Further again using (19) we have

$$\int_0^1 D(q_n) \operatorname{sign} D(f)\, dx = \sum_{k=0}^{m+1} \int_{x_{i_k}}^{x_{s_k}} q_n \operatorname{sign} f\, dx + \sum_{k=0}^{m} \gamma_k \sum_{i=s_k}^{i_{k+1}-1} q_n(x_i^*)\, \Delta x_i,$$

where $\gamma_k = \operatorname{sign} f$ while $x \in [x_{s_k}, x_{i_{k+1}}]$. Thus by (22)

$$\begin{aligned} &\left| \int_0^1 D(q_n) \operatorname{sign} D(f)\, dx \right| \\ &\quad = \left| \int_0^1 D(q_n) \operatorname{sign} D(f)\, dx - \int_0^1 q_n \operatorname{sign} D(f)\, dx \right| \\ &\quad = \left| \sum_{k=0}^{m} \gamma_k \sum_{i=s_k}^{i_{k+1}-1} q_n(x_i^*)\, \Delta x_i - \sum_{k=0}^{m} \gamma_k \sum_{i=s_k}^{i_{k+1}-1} \int_{x_i}^{x_{i+1}} q_n\, dx \right| \\ &\quad \leqslant \sum_{k=0}^{m} \sum_{i=s_k}^{i_{k+1}-1} \left| q_n(x_i^*)\, \Delta x_i - \int_{x_i}^{x_{i+1}} q_n\, dx \right|. \end{aligned} \tag{23}$$

Hence, using the representation

$$q_n(x) = q_n(x_i^*) + (x - x_i^*)\, q_n'(x_i^*) + \int_{x_i^*}^{x} (x - t)\, q_n''(t)\, dt,$$

we obtain

$$\begin{aligned} \left| q_n(x_i^*)\, \Delta x_i - \int_{x_i}^{x_{i+1}} q_n\, dx \right| &\leqslant \left| \int_{x_i}^{x_{i+1}} \int_{x_i^*}^{x} (x - t)\, q_n''(t)\, dt\, dx \right| \\ &\leqslant \Delta x_i \|q_n''\|_C \, \delta^2. \end{aligned} \tag{24}$$

Set $T = \max_{1 \leqslant i \leqslant n} \|\varphi_i''\|_C$. Then

$$\|q_n''\|_C = \left\| \sum_{i=1}^{n} a_i \varphi_i'' \right\|_C \leqslant T \sum_{i=1}^{n} |a_i| \leqslant c_{15}(U_n) \|q_n\|_C.$$

This together with (23) and (24) imply (21).

COROLLARY 1. *For any $q_n \in U_n$, we have*

$$\|D(f) - D(q_n)\|_1 - \|D(f)\|_1 \geqslant 2 \int_{A(f, q_n)} |D(f) - D(q_n)| \, dx - c_{16}(U_n) \|q_n\|_C \, \delta^2, \tag{25}$$

where $A(f, q_n) = \{x \in [0, 1] : 0 < D(f) < D(q_n) \text{ or } D(q_n) < D(f) < 0\}$.

Proof. By (21) we immediately obtain

$$\begin{aligned} &\|D(f) - D(q_n)\|_1 - \|D(f)\|_1 \\ &= \int_0^1 (D(f) - D(q_n))\{\text{sign}(D(f) - D(q_n)) - \text{sign}\, D(f)\} \, dx \\ &\quad - \int_0^1 D(q_n) \, \text{sign}\, D(f) \, dx \\ &\geqslant 2 \int_{A(f, q_n)} |D(f) - D(q_n)| \, dx - c_{16}(U_n) \|q_n\|_C \, \delta^2. \end{aligned}$$

LEMMA 2. *For any $g \in C[0, 1]$,*

$$|\, \|D(g)\|_1 - \|g\|_{1,\delta} | \leqslant c_{17}(f) \, \delta \omega_g(\delta). \tag{26}$$

Proof. By simple calculations we get

$$\begin{aligned} &|\, \|D(g)\|_1 - \|g\|_{1,\delta} | \\ &= \left| \sum_{k=0}^{m+1} \int_{x_{i_k}}^{x_{s_k}} |g| \, dx + \sum_{k=0}^{m} \sum_{i=s_k}^{i_{k+1}-1} |g(x_i^*)| \, \Delta x_i - \sum_{i=0}^{N} |g(x_i^*)| \, \Delta x_i \right| \\ &= \left| \sum_{k=0}^{m+1} \sum_{i=i_k}^{s_k-1} \int_{x_i}^{x_{i+1}} (|g| - |g(x_i^*)|) \, dx \right| \\ &\leqslant \omega_g(\delta) \sum_{k=0}^{m+1} (x_{s_k} - x_{i_k}). \end{aligned}$$

Hence by (17) we obtain (26).

COROLLARY 2. *Let* $p_n(f)_\delta \in Y_n(f)_\delta$. *Then*

$$\int_{A(f, p_n(f)_\delta)} |D(f) - D(p_n(f)_\delta)| \, dx \leqslant c_{18}(f, U_n) \, \delta\omega_f(\delta). \tag{27}$$

Proof. By (9) and (11),

$$\begin{aligned} \| p_n(f)_\delta \|_C &\leqslant c_{19}(U_n) \| p_n(f)_\delta \|_1 \leqslant 2c_{19}(U_n) \| p_n(f)_\delta \|_{1,\delta} \\ &\leqslant 4c_{19}(U_n) \| f \|_{1,\delta} \\ &\leqslant 4c_{19}(U_n)(\| f \|_1 + \omega_f(1)) = c_{20}(f, U_n). \end{aligned}$$

Therefore, according to (25), (26) and (13),

$$\begin{aligned} &\int_{A(f, p_n(f)_\delta)} |D(f) - D(p_n(f)_\delta)| \, dx \\ &\leqslant \tfrac{1}{2}\{\| D(f) - D(p_n(f)_\delta) \|_1 - \| D(f) \|_1\} + c_{21}(f, U_n) \, \delta^2 \\ &= \tfrac{1}{2}\{\| D(f - p_n(f)_\delta) \|_1 - \| D(f) \|_1\} + c_{21}(f, U_n) \, \delta^2 \\ &\leqslant \tfrac{1}{2}\{\| f - p_n(f)_\delta \|_{1,\delta} - \| f \|_{1,\delta}\} + c_{17}(f) \, \delta\omega_f(\delta) \\ &\quad + \tfrac{1}{2}c_{17}(f) \, \delta\omega_{p_n(f)_\delta}(\delta) + c_{21}(f, U_n) \, \delta^2 \\ &\leqslant c_{22}(f, U_n) \, \delta\omega_f(\delta). \end{aligned}$$

Thus estimation (27) is proved.

Now we are able to prove the upper bound of Theorem 2. We may assume that $\omega_f(h)$ is strictly increasing for any $0 < h \leqslant 1$, because $\omega_f(h) \leqslant \omega_f(h) + h \leqslant \{1 + 2/\omega_f(1)\} \;\; \omega_f(h)$, where $\omega_f(h) + h$ is already a strictly increasing modulus of continuity. Let $\{t_j^*\}_{j=1}^l \in (0, 1)$ be the points of change of sign of f. Evidently $l \geqslant n$. (Otherwise, by a well-known theorem, there exists a $q_n^* \in U_n \backslash 0$ with sign $q_n^* = \operatorname{sign} f$, which contradicts (22).)

Therefore for any $q_n \in U_n$, we have

$$\| q_n \|_C \leqslant c_{23}(U_n) \max_{1 \leqslant j \leqslant l} |q_n(t_j^*)| \qquad (c_{23}(U_n) \geqslant 1). \tag{28}$$

Set $t_0^* = 0$, $t_{l+1}^* = 1$, $F = \min_{0 \leqslant j \leqslant l} \max_{t_j \leqslant x \leqslant t_{j+1}} |f(x)|$, $F > 0$. Assume that δ is so small that

$$\sup_{p_n(f)_\delta \in Y_n(f)_\delta} \| p_n(f)_\delta \|_C \leqslant \min\{1, F\}. \tag{29}$$

Take an arbitrary $p_n(f)_\delta \in Y_n(f)_\delta \backslash 0$. According to (28) there exists a $\xi \in (0, 1)$ such that f changes its sign at ξ (thus $f(\xi) = 0$) and $\| p_n(f)_\delta \|_C \leqslant c_{23}(U_n) \, | p_n(f, \xi)_\delta |$.

Without loss of generality we may assume that $p_n(f, \xi)_\delta > 0$ and $f(x) > 0$ while $x \in (\xi, \eta)$, where η is the next point where f changes its sign. By (20)

$$0 < D(f) < 2\omega_f(x - \xi), \qquad x \in (\xi, \eta). \tag{30}$$

Analogously by (20), (8) and (29), we obtain

$$\begin{aligned} D(p_n(f)_\delta) &\geqslant p_n(f, \xi)_\delta - 2\omega_{p_n(f)_\delta}(x - \xi) \\ &\geqslant \frac{1}{c_{23}(U_n)} \| p_n(f)_\delta \|_C - 2c_4(U_n) \| p_n(f)_\delta \|_1 (x - \xi) \\ &\geqslant \frac{1}{c_{23}(U_n)} \| p_n(f)_\delta \|_C - c_{24}(U_n)(x - \xi), \qquad x \in (\xi, \eta). \end{aligned} \tag{31}$$

It follows from (29) that there exists $\bar{x} \in (\xi, \eta)$ such that

$$2\omega_f(\bar{x} - \xi) = \frac{1}{c_{23}(U_n)} \| p_n(f)_\delta \|_C - c_{24}(U_n)(\bar{x} - \xi). \tag{32}$$

Further, (32) implies that

$$\bar{x} \geqslant \xi + \omega_f^{-1}(c_{25}(f, U_n) \| p_n(f)_\delta \|_C) = \xi + \omega_f^{-1}(2\omega_f(h)), \tag{33}$$

where h is defined as solution of the equation

$$\| p_n(f)_\delta \|_C = \frac{2}{c_{25}(f, U_n)} \omega_f(h). \tag{34}$$

By (30), (31) and (32) we have for $x \in (\xi, \bar{x})$

$$D(p_n(f)_\delta) - D(f) \geqslant 2\{\omega_f(\bar{x} - \xi) - \omega_f(x - \xi)\} \geqslant 0; \qquad D(f) > 0.$$

Hence applying (27) and (33) we have

$$\begin{aligned} c_{18}(f, U_n)\, \delta \omega_f(\delta) &\geqslant \int_\xi^{\bar{x}} \{D(p_n(f)_\delta) - D(f)\}\, dx \\ &\geqslant 2 \int_\xi^{\bar{x}} \{\omega_f(\bar{x} - \xi) - \omega_f(x - \xi)\}\, dx \\ &\geqslant 2 \int_0^{\omega_f^{-1}(2\omega_f(h))} \{2\omega_f(h) - \omega_f(x)\}\, dx \\ &\geqslant 2 \int_0^h \{2\omega_f(h) - \omega_f(x)\}\, dx \geqslant 2\omega_f(h)\, h. \end{aligned}$$

This immediately implies that $h \leqslant c_{26}(f, U_n)\delta$. Finally, substituting this estimation in (34) we have

$$\| p_n(f)_\delta \|_C \leqslant c_{27}(f, U_n)\, \omega_f(\delta).$$

The upper bound of Theorem 2 is proved.

We shall give now a counterexample showing that estimation (7) is in general the best possible.

Consider the system of functions $\{\varphi_i\}_{i=1}^n$ spanning U_n. By a theorem proved in [2] there exist points $0 = y_0 < y_1 < \cdots < y_n < y_{n+1} = 1$ such that for any $1 \leqslant j \leqslant n$

$$\sum_{i=0}^{n} (-1)^i \int_{y_i}^{y_{i+1}} \varphi_j(x)\, dx = 0. \tag{35}$$

Let $0 < \delta < \min_{0 \leqslant i \leqslant n}(y_{i+1} - y_i)/2$ and set $a_i = y_i - \delta/4$; $b_i = y_i + 3\delta/4$ $(i = \overline{1, n})$. Evidently, we can choose the finite point set $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$ in such way that $\{x_j\}_{j=0}^{N+1} \cap (a_i, b_i) = \varnothing$ $(i = \overline{1, n})$ and $\max_{0 \leqslant i \leqslant N} \Delta x_i = \delta$. Let ω be an arbitrary modulus of continuity and define f by

$$\begin{aligned} f(x) &= (-1)^i\, \omega(x - y_i)/2, && x \in \left[y_i, \frac{y_i + y_{i+1}}{2} \right] \ (i = \overline{1, n-1}), \\ &= (-1)^i\, \omega(y_{i+1} - x)/2, && x \in \left[\frac{y_i + y_{i+1}}{2}, y_{i+1} \right] \left(i = \overline{1, n-1}\right), \\ &= \omega(y_1 - x)/2, && x \in [y_0, y_1], \\ &= (-1)^n\, \omega(x - y_n)/2, && x \in [y_n, y_{n+1}]. \end{aligned}$$

Then $c_{28}\, \omega(h) \leqslant \omega_f(h) \leqslant \omega(h)$,

$$|f(x_i^*)| \geqslant \omega\left(\frac{\delta}{4}\right) \Big/ 2 \geqslant \frac{1}{16}\, \omega(\delta), \qquad i = \overline{0, N}, \tag{36}$$

and by (35), $p_n(f) \equiv 0$. Let us prove that

$$\sup_{p_n(f)_\delta \in Y_n(f)_\delta} \| p_n(f)_\delta \|_C \geqslant \frac{\omega(\delta)}{64}. \tag{37}$$

Take an arbitrary $p_n(f)_\delta \in Y_n(f)_\delta$. We may assume that

$$\| p_n(f)_\delta \|_C < \frac{\omega(\delta)}{64}. \tag{38}$$

(In the opposite case there is nothing to prove.) By the characterization theorem of best discrete L_1-approximation (see [9, p. 74]), for any $q_n \in U_n$

$$\left| \sum_{i=0}^{N} q_n(x_i^*)\, \Delta x_i \operatorname{sign}\{f(x_i^*) - p_n(f, x_i^*)_\delta\} \right| \leqslant \sum_{i \in I} |q_n(x_i^*)|\, \Delta x_i, \tag{39}$$

where $I = \{i: f(x_i^*) = p_n(f, x_i^*)_\delta\}$. But by (36) and (38), I is empty and sign $\{f(x_i^*) - p_n(f, x_i^*)_\delta\} = \operatorname{sign} f(x_i^*)$ $(i = \overline{0, N})$. Thus it follows from (39), that for any $q_n \in U_n$

$$\sum_{i=0}^{N} q_n(x_i^*)\, \Delta x_i \operatorname{sign} f(x_i^*) = 0. \tag{40}$$

Set $\bar{q}_n = p_n(f)_\delta + \omega(\delta)/32$. Then by (38), $\|\bar{q}_n\|_C < \omega(\delta)/16$. Thus by (36), $\operatorname{sign}\{f(x_i^*) - \bar{q}_n(x_i^*)\} = \operatorname{sign} f(x_i^*)$ $(i = \overline{0, N})$. Using (40) we have by the characterization theorem that $\bar{q}_n \in Y_n(f)_\delta$. But (38) yields

$$\|\bar{q}_n\|_C \geqslant \frac{1}{32}\,\omega(\delta) - \|p_n(f)_\delta\|_C > \frac{1}{64}\,\omega(\delta); \tag{41}$$

hence (37) is verified.

The proof of Theorem 2 is complete.

Remark 4. Evidently, by a simple modification we can make f analytic and still obtain in (37) a lower bound $c\delta$. Thus further improvement of the smoothness of function does not improve in general the rate of convergence.

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